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HEAT TRANSFER AND TURBULENT DIFFUSION IN ONE-DIMENSIONAL HYDRODYNAMICS WITHOUT PRESSURE†

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A one-dimensional transient non-linear problem of continuum mechanics is considered, the possibility of an accurate analytic solution of which is later based on a general local analysis of singular solutions known as the Painlevé test. For one-dimensional non-linear hydrodynamic models without pressure, with the transfer of a passive impunity, which generalizes the well-known Burgers' model, it is shown that it is possible to reduce the problem to linear problems when the kinetic coefficients (viscosity and thermal conductivity) are equal. Using examples of their accurate solutions, the high sensitivity of the structure of shock waves with impurity fronts to the satisfaction of the law of conservation of impurity in the models is demonstrated. When it is satisfied, each steady propagating shock wave with a viscous structure of the velocity field is accompanied by an impurity soliton. When several such shock waves merge (the accurately solved problem), concentration of the impurity in one overall soliton occurs. It is shown that, when the action of time-dependent Gaussian random forces is taken into account, an additional diffusive spreading of the perturbations, with a time-dependent diffusion coefficient, is superimposed on the linearized viscous behaviour of the main models. © 1999 Elsevier Science Ltd. All rights reserved.

1. REDUCTION OF THE NON-LINEAR PROBLEM TO A LINEAR PROBLEM

The simplest dissipative model of a single-component complex medium is the model of a viscous heat-conducting liquid without pressure, while its simplest one-dimensional flows, taking the transfer of passive impurity into account, which has no inverse effect on the dynamics, are described by the following system of non-linear partial differential equations

$$\frac{du}{dt} \equiv \partial_t u + u\partial_x u = \nu\partial_x^2 u, \quad \frac{d\theta}{dt} \equiv \partial_t \theta + u\partial_x \theta = \chi\partial_x^2 \theta \tag{1.1}$$

Here $u = u(x, t)$ is the flow rate, $\theta = \theta(x, t)$ is a characteristic of the impurity (the temperature in the case of heat transfer or the concentration in the case of mass transfer in a two-component liquid), the notation $\partial_t u, \partial_x u, \ldots$ is used for the partial derivatives, and ν and χ are the coefficient of viscosity and the thermal diffusivity (or the diffusion coefficient), the non-dimensional ratio of which is the Prandtl (or Schmidt) number $\mathrm{Pr} \equiv \nu/\chi$.

The possibility of the complete integrability of this system of equations can be investigated using the Painlevé test for partial differential equations (we follow the terminology established in the foreign literature, although it is more correct, in view of the history of the investigations [1, 2], to call it the "Kovalevskaya–Painlevé test"). We will here use a local analysis of the behaviour of the singular points of the solutions of the equations in the version proposed by Weiss, Tabor and Carnevale [3]. When analysing the singular solution of the system of equations in question in the neighbourhood of the singularity manifold $\varepsilon(x, t) = 0$, $\varepsilon_x \equiv \partial\varepsilon/\partial x \neq 0$, $\varepsilon_t \equiv \partial_t\varepsilon/\partial t \neq 0$, we will represent it in the form of expansions in a Laurent series with pole singularities of finite order (the final principal parts of the Laurent expansions)

$$u(x,t) = \sum_{n=0}^{\infty} u_n \varepsilon^{n-\alpha}, \quad \theta(x,t) = \sum_{n=0}^{\infty} \theta_n \varepsilon^{n-\beta} \tag{1.2}$$

$$u_n = u_n(x,t), \quad \varepsilon = \varepsilon(x,t), \quad \theta_n = \theta_n(x,t)$$

In view of the fact that the first equation of the system, which is well known under the name of Burgers' equation, is independent of the second problem, it can initially be analysed separately. According to

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the well-known results in [3] for the Painlevé test, the solution of Burgers' equation can be represented in the form of a series containing a simple pole singularity (the equation $\alpha = 1$ and the form of the first coefficient function $u_0 = -2\nu\varepsilon_x$ is only the leading-order singular term of the expansion, already formulated in the equation)

$$u = -2\nu\frac{\varepsilon_x}{\varepsilon} + u_1 + u_2\varepsilon + \sum_{m=0}^{\infty} u_{m+3}\varepsilon^{m+2} \tag{1.3}$$

for the coefficients of which the following recursion relation must be satisfied

$$-(n-2)(n-1)\nu\varepsilon_x^2 u_n + (\partial_t - \nu\partial_x^2)u_{n-2} + (n-2)(\varepsilon_t - 2\nu\varepsilon_x\partial_x - \nu\varepsilon_{xx})u_{n-1} + \\ + \sum u_{n-m}[(m-1)\varepsilon_x u_m + \partial_x u_{m-1}] = 0, \quad u_k|_{k<0} = 0$$

In particular, for the two lowest coefficients of the series (with $n = 0$ and $n = 1$) we hence have

$$u_0 = -2\nu\varepsilon_x, \quad u_1\varepsilon_x + \varepsilon_t = \nu\varepsilon_{xx} \tag{1.4}$$

and for $n = 2$, the coefficient u_2 drops out from the recursion relation and we obtain the constraint

$$\partial_x(\varepsilon_t + u_1\varepsilon_x - \nu\varepsilon_{xx}) = 0$$

which is a consequence of the previous one.

It finally becomes clear at this stage that the local expansion of the solution considered possesses the required generality for solving the second-order differential equation, Burgers' equation: it contains two arbitrary functions $\varepsilon(x, t)$, $u_2(x, t)$ and therefore passes the Painlevé test. The general solution of the Cauchy problem for it should have exactly such a degree of arbitrariness. By virtue of the well-known Kovalevskaya (Cauchy–Kovalevskaya) theorem on the existence of an analytic solution of the Cauchy problem for differential equations [4] of these two functions (according to the number of initial conditions for the second-order equation), it is sufficient to seek any analytic solutions. The initial data completely define the coefficients of the expansions of these solutions.

Taking into account the expressions for the lowest coefficients, the recursion relation can also be rewritten, clearly distinguishing the higher coefficient, in the form

$$(n-2)(n+1)\nu\varepsilon_x^2 u_n = (n-2)(\varepsilon_t + u_1\varepsilon_x - \nu\varepsilon_{xx} - 2\nu\varepsilon_x\partial_x)u_{n-1} - 2\nu\varepsilon_x\partial_x u_{n-1} + \\ + \sum_{m=1}^{n-2} u_{n-m}[(m-1)\varepsilon_x u_m + \partial_x u_{m-1}] + (\partial_t + u_1\partial_x - \nu\partial_x^2)u_{n-2} \tag{1.5}$$

In this form, the "resonance" nature of the case $n = 2$ becomes obvious (at "resonance" the left and right sides of the relation vanish simultaneously).

Hence, for $n = 3$, taking the previous expressions (1.4) into account, we obtain

$$4\nu\varepsilon_x^2 u_3 = -2\nu(2\varepsilon_x\partial_x + \varepsilon_{xx})u_2 + (\partial_t + u_1\partial_x - \nu\partial_x^2)u_1 \tag{1.6}$$

If we set the arbitrary function u_2 equal to zero and require the additive term u_1 in the general expansion to satisfy the initial Burgers' equation

$$u_2 = 0, \quad \partial_t u_1 + u_1\partial_x u_1 - \nu\partial_x^2 u_1 = 0$$

then the third coefficient vanishes and, together with it, as can be seen from the recursion relation, all the more leading coefficients also vanish. Hence, the Laurent series is truncated at the final term

$$u_n|_{n\geqslant 2} = 0 \tag{1.7}$$

Finally, the solution of Burgers' equation is expressed in terms of two functions, which satisfy simple equations, of which one is again Burgers' equation

$$u = -2\nu\varepsilon^{-1}\varepsilon_x + u_1, \quad \varepsilon_t + u_1\varepsilon_x = \nu\varepsilon_{xx}, \quad \partial_t u_1 + u_1\partial_x u_1 = \nu\partial_x^2 u_1 \tag{1.8}$$

Together, these relations comprise the Bäcklund transformation, which enables one, from one solution of Burgers' equation, to construct the others [3]. In particular, by choosing $u_1 = 0$ one obtains a representation of the solution of Burgers' equation in terms of the solutions of the linear heat-conduction equation the well-known Cole–Hopf transformation

$$u=-2\nu\varepsilon^{-1}\varepsilon_x, \quad \varepsilon_t=\nu\varepsilon_{xx} \tag{1.9}$$

The equation for the passive impurity also allows of a solution in the form of expansions in the above-mentioned Laurent series (1.2), if the following recursion relation is satisfied, in addition to the previous one

$$-\chi(n-\beta)(n-\beta-1)\varepsilon_x^2\theta_n+\sum_m u_{n-m}[\partial_x\theta_{m-1}+(m-\beta)\varepsilon_x\theta_m]+$$
$$+(n-\beta-1)(\varepsilon_t-\chi\varepsilon_{xx}-2\chi\varepsilon_x\partial_x)\theta_{n-1}+(\partial_t-\chi\partial_x^2)\theta_{n-2}=0, \ \theta_k|_{k<0}=0$$

At the first step, with $n = 0$, it gives

$$[2\nu-\chi(\beta+1)]\beta\varepsilon_x^2\theta_0=0: \ \beta=2\,\mathrm{Pr}-1 \tag{1.10}$$

so that in the expansion for the impurity, the coefficient $\theta_0(x, t)$ remains arbitrary.

In the truncation of the series for the velocity field discussed above, such that relations (1.7) and (1.8) are satisfied, in the last recursion relation only one term with $m = n - 1$ under the summation sign in fact remains, and, if we take (1.10) into account, it takes the form of a formula relating the triples of neighbouring coefficients θ_n, θ_{n-1}, θ_{n-2}

$$n(n+1-2\,\mathrm{Pr})\varepsilon_x^2\theta_n=[(n-2\,\mathrm{Pr})(\mathrm{Pr}-1)\varepsilon_{xx}-(\mathrm{Pr}-n)2\varepsilon_x\partial_x]\theta_{n-1}+$$
$$+\chi^{-1}(\partial_t+u_1\partial_x-\chi\partial_x^2)\theta_{n-2} \tag{1.11}$$

Hence we see that, in addition to the case $n = 0$ mentioned above, the integer positive values of $n = N = 2\mathrm{Pr} - 1$ may also be "resonances" if the right-hand sides in (1.11) simultaneously vanish. Substituting these values we obtain

$$0\cdot\theta_N=\tfrac{1}{2}(1-N)(\varepsilon_{xx}+2\varepsilon_x\partial_x)\theta_{N-1}+\chi^{-1}(\partial_t+u_1\partial_x-\chi\partial_x^2)\theta_{N-2}$$

which is only possible when $N = 1 = \mathrm{Pr}$, i.e. when the kinetic coefficients are equal.

When $\mathrm{Pr} = 1$, the second "resonance" is found in fact at $n = 1$, which is obvious from the equation

$$2(1-\mathrm{Pr})\varepsilon_x^2\theta_1=(1-\mathrm{Pr})[(2\,\mathrm{Pr}-1)\varepsilon_{xx}-2\varepsilon_x\partial_x]\theta_0$$

The left- and right-hand sides will simultaneously vanish and the coefficient function $\theta_1(x, t)$ remains arbitrary (the second arbitrary function). We thereby also obtain the required generality (two arbitrary functions) of the solution of the equation for the impurity in the form of expansions (1.2).

For the equal kinetic coefficients, recursion relation (1.11) reduces to the following relation in the special case when $n = 2$

$$2\varepsilon_x^2\theta_2=-2\varepsilon_x\partial_x\theta_1+\chi^{-1}(\partial_t+u_1\partial_x-\chi\partial_x^2)\theta_0$$

It can be seen that the coefficient function $\theta_2(x, t)$ vanishes when θ_1 vanishes, and θ_0 obeys an equation for the impurity with convectional velocity u_1 of the initial type (1.2). Here, according to recursion relation (1.11), all the subsequent coefficients vanish, so that the Laurent series for the impurity

$$\theta_n|_{n\geqslant 1}=0$$

is truncated.

Finally, in the case of hydrodynamics with the equal dissipative coefficients, we obtain the set of relations

$$u=-2\nu\partial_x\ln\varepsilon+u_1, \quad \theta=\theta_0/\varepsilon+\mathrm{const}$$

$$e_t+u_1\varepsilon_x=\nu\varepsilon_{xx}, \quad \partial_t u_1+u_1\partial_x u_1=\nu\partial_x^2 u_1, \quad \partial_t\theta_0+u_1\partial_x\theta_0=\nu\partial_x^2\theta_0$$

which enables us, from any special solutions of the system of non-linear equations (1.1), to construct others (the Bäcklund transform). In particular, by choosing the trivial initial solution $u_1 = 0$, we obtain a representation of the solutions in the form

$$u=-2\nu\partial_x \ln\varepsilon,\quad \theta=\theta_0/\varepsilon,\quad \partial_t\varepsilon=\nu\partial_x^2\varepsilon,\quad \partial_t\theta_0=\nu\partial_x^2\theta_0 \tag{1.12}$$

so that by a replacement of the unknowns (the Cole–Hopf and Hirota transformations [5]) we can reduce the solution of the initial non-linear equations with equal kinetic coefficients to the solution of the linear heat-conduction equation.

In reality, laminar flows of air and also many turbulent flows of different liquids in the gradient approximation of turbulent fluxes of momentum and heat are characterized by Prandtl numbers close to unity.

The solution of the problem with initial data

$$u|_{t=0}=U(x),\quad \theta|_{t=0}=\Theta(x) \tag{1.13}$$

for the initial non-linear equations (1.1), when the dissipative coefficients are equal, now reduces to the solution of linear heat-conduction equations from (1.12) with initial data

$$\varepsilon|_{t=0}=\exp\left(\frac{1}{2\nu}\int_x^\infty U(\xi)d\xi\right),\quad \theta_0|_{t=0}=\Theta(x)\varepsilon|_{t=0}$$

which, using Green's function of the heat-conduction equation ($H(t)$ is the Heaviside unit function)

$$(\partial_t-\nu\partial_x^2)G=\delta(x)\delta(t):\quad G(x,t)=H(t)D(x,t)=\frac{H(t)}{2\sqrt{\pi\nu t}}\exp\left(-\frac{x^2}{4\nu t}\right) \tag{1.14}$$

can be written in general form in terms of integral convolutions of the initial distribution with the "fundamental solution" of the homogeneous equation $D(x, t)$

$$\varepsilon(x,t)=\int_{-\infty}^{+\infty}D(x-y,t)\varepsilon(y,t=0)dy=\int_{-\infty}^{+\infty}D(x-y,t)\exp\left(\frac{1}{2\nu}\int_y^\infty dzU(z)\right)dy$$

$$\theta_0(x,t)=\int_{-\infty}^{+\infty}D(x-y,t)\Theta(y)\varepsilon(y,t=0)dy=\int_{-\infty}^{+\infty}D(x-y,t)\Theta(y)\exp\left(\frac{1}{2\nu}\int_y^\infty dzU(z)\right)dy$$

In the example of the problem of the formation of the shock-wave structure and a thermal front, initially having the form of discontinuous distributions of the velocity and concentration of the passive impurity

$$U(x)=2u_0H(-x),\quad \Theta(x)=\Theta_0H(-x) \tag{1.15}$$

the result can be expressed in terms of the probability integral

$$\begin{gathered}2\varepsilon(x,t)=\operatorname{erfc}(-\xi)+\exp[-k(x-u_0t)]\operatorname{erfc}(\xi-k\sqrt{\nu t})\\ 2\theta_0(x,t)=\Theta_0\exp[-k(x-u_0t)]\operatorname{erfc}(\xi-k\sqrt{\nu t}),\quad \xi\equiv x/(2\sqrt{\nu t}),\quad k=u_0/\nu\end{gathered} \tag{1.16}$$

These distributions tend, with time, to an exponential limit, describing a simple structure of the velocity and temperature fields in uniformly propagating shock waves with a viscous transition layer

$$\begin{gathered}\varepsilon=1+\exp(-kX),\quad \theta_0=\Theta_0\exp(-kX)\\ u=\frac{2u_0}{1+\exp(kX)}=u_0\left(1-\operatorname{th}\frac{kX}{2}\right),\quad \theta=\frac{\Theta_0}{1+\exp(kX)},\quad X\equiv x-u_0t\end{gathered} \tag{1.17}$$

In the example of the problem of the evolution of the heat release, initially concentrated at a certain point ($\delta(x-x_0)$ is the Dirac delta function)

$$\theta|_{t=0}=\Theta(x)=Q_0\delta(x-x_0) \tag{1.18}$$

in the velocity field of a uniformly moving shock wave of the form (1.17) we obtain for the temperature changes

$$\theta(x,t)=\frac{\theta_0}{\varepsilon}=Q_0D(x-x_0,t)\frac{\varepsilon(x_0,t=0)}{\varepsilon(x,t)}=$$
$$=\frac{Q_0}{2\sqrt{\pi\nu t}}\frac{1+\exp(-kx_0)}{1+\exp[-k(x-u_0t)]}\exp\left[-\frac{(x-x_0)^2}{4\nu t}\right] \tag{1.19}$$

Consequently, diffusive spreading of the initially concentrated thermal perturbation occurs, with a certain additional deformation of the shock-wave flow field. No equilibrium balance between the diffusive spreading and convective transfer is reached here.

2. FLOW AND HEAT TRANSFER WHEN THERE ARE RANDOM FORCES

For the flow of a liquid, made turbulent by a random external force, the simplest case for an analytic consideration is the case when the dependence of the force on the spatial coordinates can be neglected. This means that the analysis is confined to turbulent pulsations of a smaller scale than the scale corresponding to the correlation radius of the random force.

The problem of solving inhomogeneous equations of one-dimensional hydrodynamics with a time-dependent external force

$$\partial_t v+v\partial_x v-\nu\partial_x^2 v=f(t),\quad \partial_t T+v\partial_x T-\chi\partial_x^2 T=0 \tag{2.1}$$

by replacement of the variables

$$v(x,t)=u_0(t)+u(x-x_0(t),t),\ T(x,t)=\theta(x-x_0(t),t),\ \partial_t x_0=u_0,\ \partial_t u_0=f(t)$$

reduces to solving the problem for the initial homogeneous equations (1.1). Hence, the solution of the inhomogeneous partial differential equations is reduced to solving the same homogeneous equations, the solution of ordinary differential equations for $x_0(t)$ and $u_0(t)$ and a simple replacement of variables (the generalized Galilean transformation).

For a Gaussian delta-correlated random force with zero mean

$$\langle f(t)\rangle=0,\quad \langle f(t_1)f(t_2)\rangle=f_0^2\delta(t_1-t_2) \tag{2.2}$$

for $x_0(x)$ and $u_0(t)$, due to the linear relation with this force, we will also have a Gaussian random behaviour with characteristics

$$\langle x_0(t)\rangle=0,\quad \langle u_0(t)\rangle=0$$
$$\langle x_0^2(t)\rangle=f_0^2t^3/3\equiv 2\tau,\quad \langle u_0^2(t)\rangle=f_0^2t,\quad \langle x_0(t)u_0(t)\rangle=f_0^2t^2/2$$

Due to the Gaussian form, the higher moments can be expressed in terms of the second moments and, in particular

$$\langle x_0^n(t)\rangle=\delta_{n,2m}(2m-1)!!\langle x_0^2\rangle^m,\quad \langle\exp(ax_0)\rangle=\exp\frac{a^2\langle x_0^2\rangle}{2} \tag{2.3}$$

Using the last formula and the expression for the shift operator we can easily average the velocity and temperature fields

$$\langle v(x,t)\rangle=\langle u(x-x_0(t),t)\rangle=\exp\left(\frac{\langle x_0^2\rangle}{2}\partial_x^2\right)u(x,t)$$
$$\langle T(x,t)\rangle=\langle\theta(x-x_0(t),t)\rangle=\exp\left(\frac{\langle x_0^2\rangle}{2}\partial_x^2\right)\theta(x,t) \tag{2.4}$$

It follows from this that, due to the action of the time-dependent Gaussian random force, an additional universal diffusive spreading of the average perturbations of both types (the same for the velocity and temperature fields even when their kinetic coefficients are different) is imposed on the usual diffusion of the fields in a viscous heat-conducting liquid, and is described by the same linear parabolic differential operator

$$(\partial_\tau - \partial_x^2)\langle v\rangle = 0, \quad (\partial_\tau - \partial_x^2)\langle T\rangle = 0, \quad \tau \equiv 1/2\langle x_0^2(t)\rangle \tag{2.5}$$

If we consider, as an example, the action of a random force on a shock wave with a front of the form (1.17), then, to measure the average inclination of the wave, we obtain, by virtue of (2.4), using a Fourier expansion

$$\partial_x\langle v(x,t)\rangle = -\frac{ku_0}{2}\exp\left(\frac{\langle x_0^2\rangle}{2}\partial_x^2\right)\sec h^2\left(\frac{kX}{2}\right) = -\frac{u_0}{k}\int_{-\infty}^{+\infty}\frac{q}{\operatorname{sh}(\pi q/k)}\exp(-q^2\tau + iqX)dq$$

The asymptotic form for long times

$$\partial_x\langle v\rangle \approx -\frac{u_0}{\sqrt{\pi\tau}}\exp\left(-\frac{X^2}{4\tau}\right)$$

shows that the inclination of the shock wave in the final stage falls in proportion to $\tau^{-1/2}$, i.e. as $t^{-3/2}$. The behaviour of the thermal front in the model considered is similar.

3. ONE-DIMENSIONAL HYDRODYNAMICS TAKING THE CONSERVATION OF IMPURITY INTO ACCOUNT

We will now somewhat change the initial system of equations (1.1) so that the equation for the impurity concentration allows for the law of conservation of the total amount of material

$$\partial_t u + u\partial_x u = \nu\partial_x^2 u, \quad \partial_t c + \partial_x(uc) = \chi\partial_x^2 c: \ \partial_t \int c\,dx = 0 \tag{3.1}$$

This system of equations, for equal kinetic coefficients ($\nu = \chi$), can also be transformed to linear equations using a similar Cole–Hopf–Hirota transformation (the system of equations considered is connected with system (1.1) by replacing the initial variable $c = \partial\theta/\partial x$)

$$u = -2\nu\partial_x \ln\varepsilon, \quad c = \partial_x(\theta_0/\varepsilon), \quad \partial_t\varepsilon = \nu\partial_x^2\varepsilon, \quad \partial_t\theta_0 = \nu\partial_x^2\theta_0 \tag{3.2}$$

The solution of Eqs (3.1) with initial data

$$u|_{t=0} = \frac{2u_0}{1+\exp(kx)}, \quad c|_{t=0} = Q_0\delta(x - x_0) \tag{3.3}$$

i.e. the solution of the problem of the diffusion of an initially concentrated impurity in a flow of liquid, related to a uniformly moving structurized shock wave, according to (3.2) has a form somewhat different from (1.19)

$$c = \partial_x\frac{\theta_0}{\varepsilon} = Q_0\partial_x\int_{x_0}^{\infty} D(x-y,t)\frac{\varepsilon(y,t=0)}{\varepsilon(x,t)}dy = \frac{Q_0}{2\sqrt{\pi\nu t}}\frac{1+\exp(-kx_0)}{1+\exp(-kX)}\exp\left[-\frac{(x-x_0)^2}{4\nu t}\right] +$$
$$+\frac{kQ_0}{8}\left[\operatorname{erfc}\frac{x_0 - x}{2\sqrt{\nu t}} - \operatorname{erfc}\frac{x_0 - x + 2u_0 t}{2\sqrt{\nu t}}\right]\operatorname{sech}^2\left(\frac{kX}{2}\right) \tag{3.4}$$

This solution consists of two groups of terms. The first is identical with expression (1.19), obtained when solving the same initial problem for system of equations (1.1) and which describes the complete attenuation of the initial perturbation when it spreads diffusely in space. The asymptotic form for long times for the second group of terms is a steady propagating distribution of impurity of the solitary-wave type

$$c(x,t) = \frac{1}{4}kQ_0\operatorname{sech}^2\frac{k(x-u_0t)}{2} \tag{3.5}$$

Such a soliton, in accordance with the conservation of material, for a description by Eqs (3.1), steadily transfers the quantity of impurity Q_0, i.e. a balance occurs between the diffusive spreading of the impurity and its concentration for transfer by the flow field of the shock wave, unlike the case of solution (1.19) for model for (1.1). The steadily propagating shock wave of the type (1.17) accompanies the impurity soliton of the form (3.5). An accurate solution of the problem of the merging of several shock waves is obtained in the same way by representing the solution of the linear heat-conduction equation for $\varepsilon(x, t)$ by the sum of the exponential contributions from each wave [6]. Impurity solitons, which accompany the shock wave, when they merge, form a higher and narrower rapidly moving soliton together with an overall shock wave. Such an impurity concentration occurs in accordance with the conservation of the total amount of impurity.

4. MULTIDIMENSIONAL GENERALIZATION

The approach developed above allows of a simple formal extension to the case of a multidimensional model of a medium without pressure, when the liquid flow can be assumed to be potential flow

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=\nu\nabla^2\mathbf{u},\quad \frac{\partial\theta}{\partial t}+\mathbf{u}\nabla\theta=\chi\nabla^2\theta$$

This system, when the kinetic coefficients are equal, can be reduced, by a Cole–Hopf–Hirota transformation

$$\mathbf{u}=-2\nu\nabla\ln\varepsilon,\quad \theta=\frac{\theta_0}{\varepsilon}$$

irrespective of the dimension, to a pair of linear heat-conduction equations

$$\partial_t\varepsilon=\nu\nabla^2\varepsilon,\quad \partial_t\theta_0=\nu\nabla^2\theta_0$$

If we also bear in mind the potential external forces $\mathbf{f}(r, t) = -\nabla(\mathbf{r}, t)$, this replacement for equations with the equal kinetic coefficients as before, remains effective, but the final linear equation becomes an equation with a variable coefficient

$$\partial_t\varepsilon=\nu\nabla^2\varepsilon+\frac{1}{2\nu}U(\mathbf{r},t)\varepsilon,\quad \partial_t\theta_0=\nu\nabla^2\theta_0+\frac{1}{2\nu}U(\mathbf{r},t)\theta_0$$

If the external forces are random, this enables the algebraically non-linear stochastic hydrodynamic problem to be reduced to a much simpler linear algebraic problem, although the stochastic non-linearity is retained.

REFERENCES

1. ABLOWITZ, M. J. and SIEGUR, H., *Solitons and the Inverse Scattering Transform*. SIAM, Philadelphia, 1981.
2. KOVALEVSKAYA, S. V., The problem of the rotation of a solid about a fixed point. In *Scientific Papers*. Izd. Akad. Nauk SSSR, Moscow, 1948, 153–220.
3. WEISS, J., TABOR, M. and CARNEVALE, G., The Painlevé property for partial differential equations. *J. Math. Phys.* 1983, **24**, 522–526.
4. PETROVSKII, I. G., *Lectures on Partial Differential Equations*. Fizmatgiz, Moscow, 1961.
5. HIROTA, R., Direct methods in the theory of solitons. In *Solitons* (edited by R. Bullough and P. J. Caudrey). Springer, Berlin, 1980.
6. WHITHAM, J. G. B., *Linear and Nonlinear Waves*. John Wiley, New York, 1974.

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